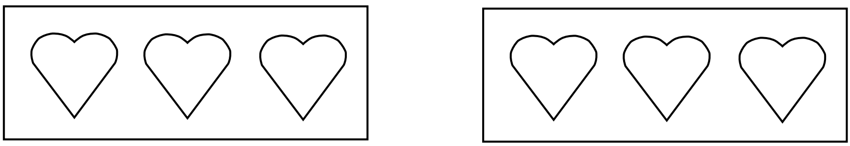
8. COMPLEX NUMBERS

§8.1. If It Doesn't Exist – Invent It

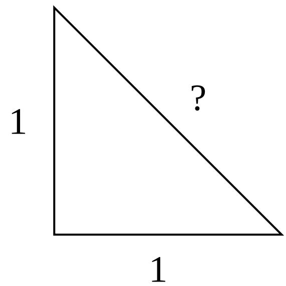
In the beginning there were the numbers 1, 2, 3, … – the so-called 'counting numbers'. Men discovered that these numbers could be combined by means of operations of addition, subtraction, multiplication and division. These operations had a practical significance. For example multiplication corresponded to the amalgamation of equal-sized heaps.



2 × 3 = 6

Although algebra had not yet been invented, there were equations, such as $2x = 7$, that had no solutions. Abdul found a problem when he wanted to bequeath his 7 camels equally to his two sons. But if he had 7 large cheeses he could cut one in half. So fractions were seen to have a practical significance and soon mankind was happy with the number system of positive rationals.

But this system was found to be deficient. It was assumed that every length corresponded to a number but when Pythagoras proved his theorem there was a problem. A right angled isosceles triangle with the shorter sides having length 1 would have to have a hypotenuse whose square is 2.



Of course we would simply write the length of the hypotenuse as $\sqrt{2}$, but that's because we have a richer number system. When fractions, that is positive rational numbers, were all that existed, there was no solution to the equation $x^2 = 2$.

The argument they used was to suppose that $\left(\frac{m}{n}\right)^2 = 2$, where m, n are counting numbers (positive integers) with no common factor. If they had a common factor we would cancel down. Then $m^2 = 2n^2$. (Of course they had to carry this argument out in words because they didn't have the luxury of algebraic notation.) So m^2 is even. If m was odd, m^2 would be odd, so m must be even. Write $m = 2k$ for some whole number k. So we have $(2k)^2 = 2n^2$ which leads to $n^2 = 2k^2$. But this means that n^2, and hence n, is even. So both m and n are even, contradicting our assumption that they have no common factor.

This discovery was a great shock to the mathematicians of the time. The world no longer seemed to be perfect and tidy. One could find numbers whose square is close to 2, as close as one likes in fact. But at best they are only approximations. Gradually however the world of arithmetic recovered from the shock and new numbers were invented – irrational numbers. So the number system in which arithmetic was carried out became the system of positive real numbers, although the decimal system for representing them came much later.

Then in the 7th century the Indian mathematician, Brahmagupta, wrote about negative numbers, and even zero, though the ideas may have been around earlier. He considered a negative number as a ‘debt’ and a positive number as a ‘fortune’ and he explained many of the familiar rules in these terms.

> A debt minus zero is a debt.
> A fortune minus zero is a fortune.
> Zero minus zero is a zero.
> A debt subtracted from zero is a fortune.
> A fortune subtracted from zero is a debt.
> The product of zero multiplied by a debt or fortune is zero.
> The product of zero multiplied by zero is zero.
> The product or quotient of two fortunes is one fortune.

The product or quotient of two debts is one fortune.
The product or quotient of a debt and a fortune is a debt.
The product or quotient of a fortune and a debt is a debt.

By the 17th century, with the number system consisting of all the real numbers, mathematicians were worried about quadratic equations, such as $x^2 + x + 1 = 0$, that have no solutions. In fact methods were invented to solve problems in the calculus of real variables that used 'imaginary' numbers such as the square root of minus one. These were considered to be fictitious entities, to be handled cautiously. They didn't exist, but if you pretended they did you got real results, It's a bit like parents pretending that Father Christmas exists in order to get the real outcome of children improving their behaviour in the weeks leading up to Christmas. But gradually square roots of -1 were accepted as genuine numbers, even if they were still called 'imaginary numbers' for technical reasons.

§8.2. The Field of Complex Numbers

To solve the problem of certain quadratic equations having no solution we invent the number 'i' with the property that $i^2 = -1$. Since there is a sense in which all numbers are invented the number i exists just as much as the square root of 2, or even three. Does the number three really exist? It is an abstraction and you can't ever see it, or weigh it. But we want our enlarged system to obey all of the usual algebraic rules.

Since we want our new system to be closed under addition and multiplication we must include in our system, all imaginary numbers, that is real multiples of i, as well as all numbers that are the sum of a real and an imaginary number – that is, all numbers of the form $a + b\text{i}$ where a, b are real. We call these numbers **complex numbers**, not because they're complicated or difficult to understand, but simply because they are made up of two parts.

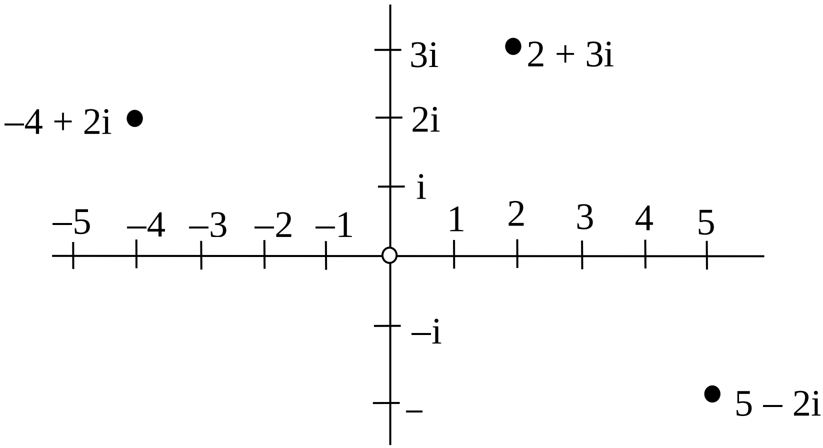
If we are asked what is the result of adding 2 and 3i we would have to answer 2 + 3i. If we were asked to simplify this we have to say 2 + 3i. In its simplified form it is made up of two components. This is not so very different to fractions such as $\frac{2}{3}$.

We can add and multiply these complex numbers by extending the associative and distributive laws to this enlarged system. For example (1 + 2i) + (3 + 4i) = 4 + 6i and

$(1 + 2i)(3 + 4i) = 3 + 6i + 4i + 8i^2 = 3 + 10i – 8 = –5 + 10i$.

§8.3. Geometrical Interpretation of Complex Numbers

Where do these new complex numbers fit on the number line? They don't! We have to branch out into 2 dimensions.

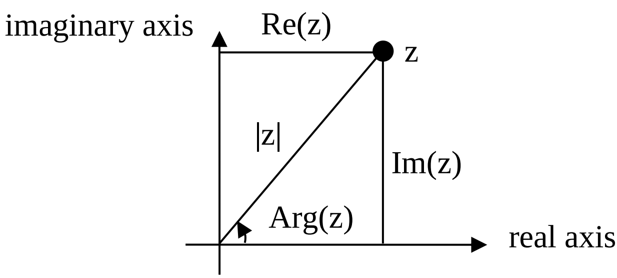


Although a complex number comes in two parts it's often convenient to denote it by a single symbol, such as z. If z = x + iy the **real part** of z is defined to be x and is denoted by **Re(z)**. The **imaginary part** of z is defined to be y and is denoted by **Im(z)**.

Note that the imaginary part is actually a real number. It's not 'iy' as one might have thought – just the coefficient of i. You may also wonder why we write 'iy' and not 'yi'. Of course, since complex numbers commute under multiplication. we're entitled to write it this way. But why we write 'a + bi' and 'x + iy' is a mystery. Perhaps subconsciously we do this because we want to write the factors in alphabetic order.

The coordinates of the point that represents a complex number are its real and imaginary parts. But there's another set of coordinates that is useful in many contexts – polar coordinates. If r is the distance of the point P(x, y) from the origin and θ is the angle between OP and the positive x-axis (O being the origin) then $x = r \cos \theta$ and $y = r \sin \theta$. The **polar coordinates** of the point are (r, θ).

If this point represents the complex number $z = x + iy$ the **modulus** of z is defined to be r and the **argument** of z is defined to be θ. We denote the modulus of z by **|z|** and the argument of z by Arg(z). Note that the modulus of a real number is simply the absolute value, which is why we use the same notation. Let us summarise.



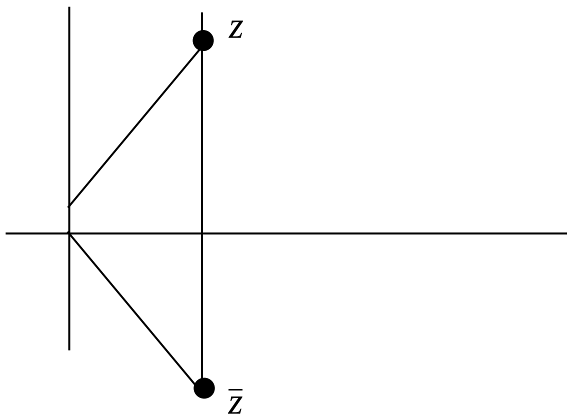
If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$ and $x = r \cos \theta$, $y = r \sin\theta$ where $r = |z|$ and $\theta = \text{Arg}(z)$.

Theorem 1: $|z_1 z_2| = |z_1|.|z_2|$
Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)$ and so

$$|z_1 z_2|^2 = (x_1 x_2 - y_1 y_2)^2 + (x_2 y_1 + x_1 y_2)^2$$
$$= x_1^2 x_2^2 + y_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 y_2^2 - 2x_1 x_2 y_1 y_2 + 2x_1 x_2 y_1 y_2$$
$$= x_1^2 x_2^2 + y_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 y_2^2$$
$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2 |z_2|^2.$$ We then take positive square roots. ✋☺

§8.4. Conjugates

The **conjugate** of the complex number $z = x + iy$ is $\bar{z} = x - iy$.
Geometrically it's the mirror image of z in the real axis.



Theorem 2: $z\overline{z} = |z|^2$ and $z + \overline{z} = 2\,\mathrm{Re}(z)$.
Proof: If $z = x + iy$ then

$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$ and

$z + \overline{z} = 2x = 2\,\mathrm{Re}(z)$.✋☺

Theorem 3: For all complex numbers z_1, z_2, z:
(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;
(2) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;
(3) $\overline{\overline{z}} = z$;
(4) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{z}{\overline{z}}$
Proof: Left as an exercise. ✋

Example 3: $\dfrac{1}{3 - 2i} = \dfrac{3 + 2i}{13}$.

Theorem 4: If the complex number z is a solution to the polynomial equation
$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ and $a_0, a_1, \ldots, a_n$ are all real then $\overline{z}$ is also a solution.
Proof: Suppose $a_n z^n + \ldots + a_1 z + a_0 = 0$.
Then $\overline{a_n z^n + \ldots + a_1 z + a_0} = \overline{0} = 0$.
Hence if the a_i are real $a_n(\overline{z})^n + \ldots + a_1(\overline{z}) + a_0 = 0$. ✋☺

§8.5. De Moivre's Theorem

If $z = x + iy$ and the modulus of z is r and the argument is θ then $z = r\cos\theta + i\, r\sin\theta$

$= r(\cos\theta + i\sin\theta)$.

Sometimes we abbreviate this to $z = r\,\text{cis}\theta$. This is called the **polar form**.

Theorem 5: $\text{cis}(\theta_1 + \theta_2) = \text{cis}\theta_1.\text{cis}\theta_2$.

Proof: $\text{cis}(\theta_1 + \theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \text{cis}\theta_1.\text{cis}\theta_2$. 🖐☺

So cisθ behave like a power. In fact it can be written as $e^{i\theta}$ where e is the mathematical constant whose approximation is 2.718… The complete justification for this notation is outside the scope of this book, but if you know some calculus and you're prepared to skip over some details the following explanation will probably be sufficient for you.

$\text{cis}\theta = \cos\theta + i\sin\theta$.

Thus $\frac{d}{d\theta}\text{cis}\theta = -\sin\theta + i\cos\theta$

$= i(\cos\theta + i\sin\theta)$

$= i\,\text{cis}\theta$.

And so $\frac{1}{\text{cis}\theta}\left(\frac{d}{d\theta}\text{cis}\theta\right) = i$.

Hence $\frac{d}{d\theta}\log_e \text{cis}\theta = i$.

Integrating, $\log_e \text{cis}\theta = i\theta + c$ and so $\text{cis}\theta = e^{i\theta+c}$.

But $\text{cis}0 = 1 = e^0$ and so $c = 0$.

Hence $\text{cis}\theta = e^{i\theta}$. ✋

Differentiating and integrating functions of a complex variable need to be properly defined and the extension of the rules for the calculus of real-valued functions to complex-valued ones need to be carefully proved before this could be called a rigorous proof. Still, it will do for now.

Theorem 6 (De Moivre's Theorem):

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all integers n.

Proof: For $n > 0$ this follows from Theorem 5.

For $n = 0$ both sides are equal to 1.

If $n < 0$ and $m = -n$ then:

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos(m\theta) + i\sin(m\theta)}$$

$$= \frac{\cos(m\theta) - i\sin(m\theta)}{1}$$

$$= \cos(n\theta) + i\sin(n\theta).$$ ✋☺

Among other things, De Moivre's Theorem is useful for expressing cos(nθ) and sin(nθ) in terms of cosθ and sinθ.

Example 4: Express cos3θ in terms of cosθ and sinθ.
Solution: Put c = cosθ and s = sinθ.
Then $\cos 3\theta + i\sin 3\theta = (c + is)^3 = c^3 + 3ic^2s + 3i^2cs^2 + i^3s^3$
$= c^3 - 3cs^2 + i(3c^2s - s^3)$
by the Binomial Theorem.
Equating real parts we get $\cos 3\theta = c^3 - 3cs^2$.
If we want to we can express this entirely in terms of c = cosθ we write
$\cos 3\theta = c^3 - 3c(1 - c^2) = 4c^3 - 3c$.

Example 5: Express sin5θ in terms of s = sinθ.
Solution: $\cos 5\theta + i\sin 5\theta = (c + is)^5 = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$.
Equating imaginary parts, $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$
$= 5(1 - s^2)^2s - 10(1 - s^2)s^3 + s^5$
$= 5(1 + s^4 - 2s^2) - 10(s^3 - s^5) + s^5$
$= 16s^5 - 20s^3 + 5s$.

§8.6. *n*-th Roots of a Complex Number

What are the square roots of i? If the equation $z^2 - i = 0$ had no solutions we'd be off again, inventing square roots of i and extending the complex number system yet again. In fact no further extension is necessary since every non-constant polynomial equation has a solution in the field of complex numbers. This is known as the **Fundamental Theorem of Algebra**. We won't prove it here as it is a fairly deep theorem and requires some non algebraic techniques.

A particular case of the Fundamental Theorem of Algebra is the existence of n-th roots for any complex number, for all $n \geq 1$.

Theorem 7: The n-th roots of r cisθ, for $r > 0$, are $r^{1/n}\text{cis}\left(\frac{\theta + 2k\pi}{n}\right)$ for $k = 0, 1, \ldots, n - 1$.

Proof: It's easy to check that these are indeed solutions. Since a polynomial of degree n can have no more than n distinct zeros, we have found them all. ✋☺

But another approach is to suppose that $z^n = r$ cisθ and to write $z = \rho$ cisφ.

Then $z^n = \rho^n\text{cis}(n\varphi) = r$ cisθ and so $\rho^n = r$ and $n\varphi = \theta + 2k\pi$. (If a complex number is written as r cisθ, for $r > 0$, there's no guarantee that θ is the argument – merely that θ differs from the argument by a multiple of 2π.)

Hence $\rho = r^{1/n}$ and $\varphi = (\theta + 2k)/n$. The values $k = 0, 1, \ldots ,$ $n - 1$ give the n distinct possibilities.

Example 6: Find the three cube roots of 8.

Solution: 8 = 8cis0 so the cube roots are $8^{1/3}$cis0, $8^{1/3}$cis(2π/3), $8^{1/3}$cis(4π/3), that is, 2, 2cis(2π/3), 2cis(4π/3). We can put these in $x + iy$ form using the fact that cos(2π/3) = cos(4π/3) = −1/2 and sin(2π/3) = √3/2 and sin(4π/3) = −√3/2. So the cube roots of 8 are:

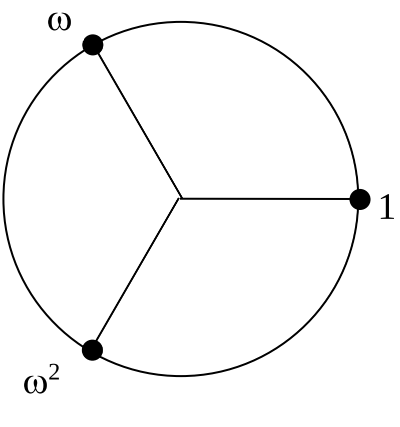
$$2, \quad 2\left(\frac{-1+\sqrt{3}\,i}{2}\right), \quad 2\left(\frac{-1-\sqrt{3}\,i}{2}\right),$$

that is, 2 and – 1± √3i. Notice that since the polynomial $z^3 - 8$ has real coefficients the non-real cube roots come as a conjugate pair.

The non-real cube roots of 1 are $\left(\frac{-1\pm\sqrt{3}\,i}{2}\right)$. They often occur
in mathematics and so are given special symbols:

$\omega = \text{cis}(2\pi/3) = \left(\frac{-1+\sqrt{3}\,i}{2}\right)$ and

$\omega^2 = \text{cis}(4\pi/3) = \left(\frac{-1-\sqrt{3}\,i}{2}\right)$.



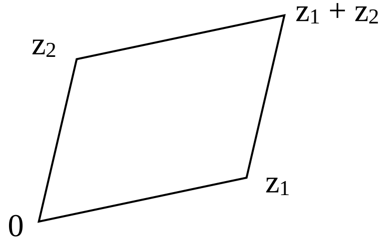
Note that $1 + \omega + \omega^2 = 0$.

More generally, the n-th roots of 1 are $\text{cis}(2k\pi/n)$ for $k = 0, 1, \ldots, n-1$.
If $\rho = \text{cis}(2\pi/n)$ we can write these as $1, \rho, \rho^2, \ldots, \rho^{n-1}$.
Notice that they are evenly spaced around the unit circle.
Note too that since the sum of the zeros of a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is $-a_{n-1}/a_n$ it follows that for $n \geq 2$ we have $1 + \rho + \rho^2 + \ldots + \rho^{n-1} = 0$.
Alternatively we can sum the GP to get $\dfrac{\rho^n - 1}{\rho - 1} = 0$ since $\rho^n = 1$.

Example 7: Find the fifth roots of $16 + 16\sqrt{3}\,i$.
Prove that one of them is twice a cube root of -1.
Solution: $16(1 + \sqrt{3}\,i) = 32\left(\dfrac{1 + \sqrt{3}\,i}{2}\right) = 32\text{cis}\left(\dfrac{\pi}{3}\right)$.
The fifth roots are thus $2\text{cis}\left(\dfrac{\pi/3 + 2k\pi}{5}\right)$ for k = 0, 1, 2, 3, 4.
These are $2\text{cis}(\pi/15)$, $2\text{cis}(7\pi/15)$, $2\text{cis}(13\pi/15)$, $2\text{cis}(19\pi/15)$, $2\text{cis}(25\pi/15)$.
The last of these is $2\text{cis}(5\pi/3)$.
Since $[\text{cis}(5\pi/3)]^3 = \text{cis}(5\pi) = \text{cis}\pi = -1$, the last claim has been verified.

§8.7. The Geometry of Complex Numbers

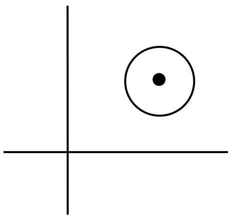
Theorem 8: If a parallelogram is formed from the three points 0, z_1 and z_2, the fourth point is $z_1 + z_2$. ✋



Theorem 9: $z_1 - z_2$ represents a point such that the directed line segment from 0 to it has the same length and direction as the directed line segment from z_2 to z_1. ✋

Corollary: $|z_1 - z_2|$ is the distance between z_1 and z_2.

Corollary: $|z - z_0| = r > 0$ is the equation of a circle with centre z_0 and radius r.



Theorem 10: The set of points $z = (1 - \lambda)z_1 + \lambda z_2$ is the straight line through z_1 and z_2. ✋

Theorem 11: The point dividing the directed line segment joining z_1 to z_2 in the ratio k:1 is

$z = \dfrac{z_1 + kz_2}{1 + k}$. ☟

Corollary: The midpoint of z_1 and z_2 is $\dfrac{z_1 + z_2}{2}$

Example 8: Prove that the set of points in the complex plane satisfying the equation
$(z - 2)^5 = (z + i)^5$ lie on a straight line. Find its equation.

Solution: Suppose $(z - 2)^5 = (z + i)^5$. Then $\left(\dfrac{z - 2}{z + i}\right)^5 = 1$.

Hence $\left|\dfrac{z - 2}{z + i}\right|^5 = 1$ and so $\left|\dfrac{z - 2}{z + i}\right| = 1$. So $|z - 2| = |z + i|$.

Thus the distance of z from 2 is the same as its distance from –i. It therefore lies on the perpendicular bisector of the line joining 2 to –i.

By elementary coordinate geometry these points have coordinates (2, 0) and (0, –1).

The slope of this line is ½ and so the perpendicular bisector has slope –2.

It passes through the midpoint of the line joining (2, 0) and (0, –1) which is (1, – ½).

So the equation is $y + \frac{1}{2} = -2(x - 1)$, which can be simplified to $4x + 2y = 3$.

§8.8. Application of Complex Numbers to Alternating Current

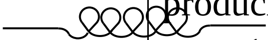
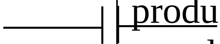
Complex numbers are an important tool in electronics. The following brief sketch of the relevant physics serves to enable the reader with no previous knowledge of electronics to understand the worked example that follows.

When an electric current flows through a wire the two quantities used to describe the situation are voltage and current. **Current** is the measure of the rate at which the electricity is flowing and **voltage** is a measure of the force with which it is being pushed along. Current is measured in units called amps (1 amp is a certain amount of electricity flowing per second). and voltage is measured in units called volts.

An electrical circuit is a path through which electricity can flow. On its journey around the circuit electrical energy can be used up in one of three ways:

(i) heating the wire;
(ii) producing a magnetic field;
(iii) producing an electric field.

There are components of electrical circuits in which the energy is dissipated in these three ways. There are resistors, inductors and capacitors and their ability to use up electrical energy in these ways is called **resistance**, **inductance** and **capacitance** respectively. These components are described in the following table.

Component	**Symbol**	**Energy used up by**	**Measurement**
Resistor		heating the wire	resistance = R ohms
Inductor		producing a magnetic field	inductance = L henrys
Capacitor		producing an electric field	capacitance = C farads

With alternating current the flow of current and voltage ebb and flow, usually very rapidly. The relationships between voltage and current and time have the following form:

$$E = E_0 \cos(2\pi ft + \alpha)$$
$$I = I_0 \cos(2\pi ft + \beta)$$

where E and I denote voltage and current respectively, where *f* denotes the frequency (in cycles per second) and *t* denotes time (in seconds). E_0 and I_0 are the maximum values of E and I and α and β are constants.

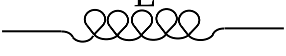
It will be noticed that E and I can reach their peaks at different times. This difference is called the **phase difference** and E and I are said to be **in phase** when the phase difference is zero.

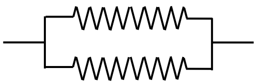
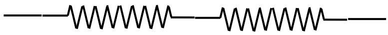
The technique of using complex numbers to handle alternating current is to replace $E = E_0\cos(2\pi ft + \alpha)$ by $E = E_0\text{cis}(2\pi ft + \alpha)$ or to use the notation most commonly used in this context $E = E_0\, e^{(2\pi ft + \alpha)i}$. Similarly we write $I = I_0\, e^{(2\pi ft + \alpha)i}$.

The actual voltage and current are the real parts of E and I, but it's convenient to work with them as complex variables.

The basic formula is the one that connects E and I for a circuit involving a single resistor. It is E = IR where E is the voltage in volts, I is the current in ohms and R is the resistance in ohms.

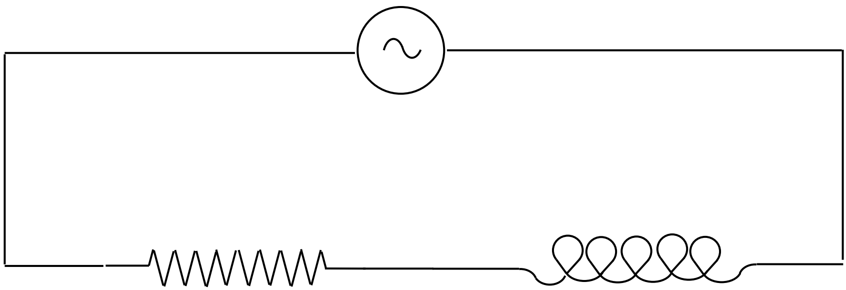
A more complicated circuit can be investigated by replacing it by a single resistor and then applying the above equation. Inductors and capacitors can be treated as resistors with imaginary resistances.

Part of circuit	**Regard as a resistor with resistance**
L  inductor	$2\pi fLi$
C  capacitor	$\dfrac{i}{2\pi fC}$

R_1  R_2 resistors in parallel	$$\frac{1}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}$$
R_1 R_2  resistors in series	$R_1 + R_2$

Example 9:

$$E = 350e^{300it} \text{ (volts)}$$



40 ohms 0.1 henrys

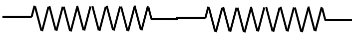
What is the current flowing when the voltage is a maximum?

NOTES: The symbol  denotes the source of the alternating current. The maximum voltage here is 350 volts and the frequency is $\frac{300}{2\pi}$ which is approximately 50 cycles per second. This is reasonably close to the domestic power in many countries. (The figure of 240 volts that's normally quoted is the 'root mean square' value, a certain type of average value.)

Solution: $2\pi f = 300$

Replace  by 

0.1 $300 \times 0.1 \times i = 30i$

Replace  by 

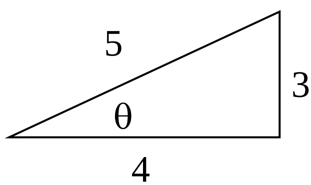
40 30i 40 + 30i

$$I = \frac{E}{R} = \frac{350e^{300it}}{40 + 30i}.$$

We now express 40 + 30i in polar form.

$|40 + 30i| = \sqrt{2500} = 50.$

Thus $40 + 30i = 50e^{i\theta}$ where θ is obtained from the following diagram.

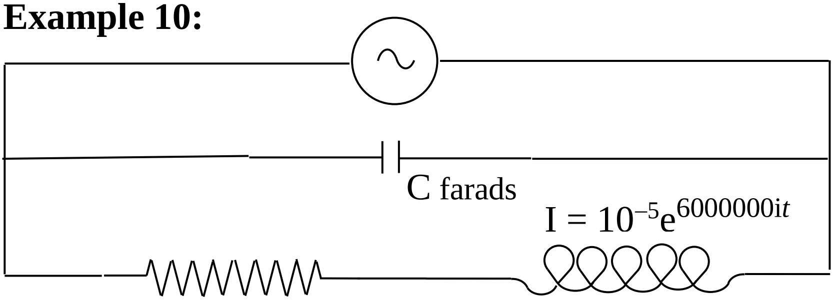


$$\text{Thus } I = \frac{350e^{300it}}{50e^{i\theta}} = 7e^{i(300t-\theta)}.$$

The real part of E is $350\cos 300t$ which attains a maximum at $t = 0$. The value of I at $t = 0$ is $7e^{-i\theta}$. The real part is

$7\cos(-\theta) = 7\cos\theta = 7\left(\frac{4}{5}\right) = 5.6$ so the current, when the voltage is a maximum, is 5.6 amps.

Example 10:



Find the value of C for which the voltage is greatest.
If C is fixed at this value and I is changed to $0.00001e^{5900000it}$ find the new value of the voltage as a percentage of the previous value.

NOTES: This illustrates the basic principal behind the tuning of an AM radio receiver. The frequency here is $\frac{6 \times 10^5}{2\pi}$ which is approximately 950,000 cycles per second, that is 950kHz. This is the broadcasting frequency of one of the popular Sydney commercial radio stations, "950 on your portable radio". Tuning is done by adjusting the variable capacitor so that the voltage is greatest for this frequency.

The second value of I corresponds to another radio station broadcasting on a nearby frequency of 939kHz.

Signals from this station will still be picked up but the corresponding voltage is considerably less than for the signal to which the receiver is tuned.

Solution: Consider the general frequency f. Put $\omega = 2\pi f$ and $I = 10^{-5}e^{i\omega t}$. The capacitor is equivalent to a resistance of $-\dfrac{i}{\omega C}$ and the inductor is equivalent to a resistance of $4 \times 10^{-6}\omega i$. Thus the whole circuit is equivalent to a resistance R where

$$\frac{1}{R} = \frac{\omega C}{-i} + \frac{1}{0.1 + 4 \times 10^{-6}\omega i}$$

$$= \omega C i + \frac{0.1 - 4 \times 10^{-6}\omega i}{16 \times 10^{-12}\omega^2 + 0.01}$$

$$= \frac{0.1}{16 \times 10^{-12}\omega^2 + 0.01} + i\left(\omega C - \frac{4 \times 10^{-6}\omega}{16 \times 10^{-12}\omega^2 + 0.01}\right).$$

Thus

$$|R^{-1}|^2 = \left(\frac{0.1}{16 \times 10^{-12}\omega^2 + 0.01}\right)^2 + \left(\omega C - \frac{4 \times 10^{-6}\omega}{16 \times 10^{-12}\omega^2 + 0.01}\right)^2.$$

If we now let R_1 denote the value of R when $\omega = 6{,}000{,}000$ and R_2 the value when $\omega = 5{,}900{,}000$ then

$$|R_1^{-1}|^2 = \left(\frac{0.1}{576.01}\right)^2 + \left(6 \times 10^6 C - \frac{24}{576.01}\right)^2 \text{ and}$$

$$|R_2^{-1}|^2 = \left(\frac{0.1}{556.97}\right)^2 + \left(5.9 \times 10^6 C - \frac{4 \times 5.9}{556.97}\right)^2.$$

Now for any ω, if $R = |R|e^{i\theta}$ then $E = IR$

$= 10^{-5}|R|\, e^{i(6000000t+\theta)}$.

Voltage is thus greatest when $|R|$ is greatest, that is when $|R^{-1}|^2$ is least.

For $\omega = 6000000$ this is clearly when

$$C = \frac{4}{576.01} \times 10^{-6}$$

$$= 0.0069 \text{ farads.}$$

For this value of C $|R_1^{-1}| = \dfrac{0.1}{576.01} = 1.74 \times 10^{-4}$ and

$$|R_2^{-1}| = \sqrt{\left(\frac{0.1}{556.97}\right)^2 + \left(\frac{5.9 \times 4}{576.01} - \frac{5.9 \times 4}{556.97}\right)^2} = 1.412 \times 10^{-3}.$$

If E_1 and E_2 denote the voltage for $\omega = 6000000$ and 5900000 respectively then

$$\frac{|E_2|}{|E_1|} = \frac{|R_2|}{|R_1|} = \frac{|R_1^{-1}|}{|R_2^{-1}|}$$

$$= \frac{1.74 \times 10^{-4}}{1.412 \times 10^{-3}} = 0.12 \text{ approximately.}$$

Thus when the circuit is tuned to 950 kHz the voltage for a signal at 939 kHz is only 12% as strong as for a signal of the same strength at the frequency that the circuit is tuned to receive.

EXERCISES FOR CHAPTER 8

Exercise 1: Express each of the following in the x + iy form:

(a) (2 + 3i) + (5 – 7i); (b) (1 + 3i)(4 – 7i); (c) $\frac{1}{1 + 3i}$;

(d) $\frac{1 - 3i}{1 + 3i}$.

Exercise 2: Express each of the following in the x + iy form:

(a) (3 – 7i) – (1 + 2i); (b) $(1 + 3i)^2$; (c) $\frac{1}{2 - 3i}$; (d) $\frac{1 + i}{1 - i}$.

Exercise 3: (i) Express –4i in the polar form.
(ii) Hence solve the equation $z^2 = -4i$.
(iii) Using the quadratic equation formula solve $z^2 - (1 + i)z + \frac{3}{2} i = 0$, giving the solutions in the x + iy form.

Exercise 4: Solve the quadratic equation $z^2 - (1 + \sqrt{3}i)z - 1 = 0$, giving the solutions in the
x + iy form.

Exercise 5: (i) Solve the equation $Z^4 = 1$;

(ii) Hence solve the equation $\left(\frac{z-1}{z}\right)^4 = 1$.

(iii) Hence solve the equation $4z^3 - 6z^2 + 4z - 1 = 0$.

Exercise 6: Solve the equation $(z + i)^3 = (2z - i)^3$.

Exercise 7: Verify that $1 + 2i$ is a zero of the polynomial $2z^3 - 7z^2 + 16z - 15$ and find the other two zeros.

Exercise 8: Verify that $1 - i$ is a zero of the polynomial $z^4 - z^3 + z^2 + 2$ and find the other three zeros.

Exercise 9: On the complex plane sketch the points z such that:
(a) $|z| = 1$; (b) $\text{Arg}(z) = \pi/4$; (c) $\text{Re}(z) > 0$; (d) $|z - i| = 2$; (e) $|z - i| = |z - 1|$.

Exercise 10: On the complex plane sketch the points z such that:
(a) $|z - 1| = 1$;
(b) $\text{Arg}(z - i) = \pi/4$;
(c) $z = \overline{z}$;
(d) $\text{Re}(z) = |z|$;
(e) $\text{Arg}(z - 1) = \text{Arg}(z - i)$.

Exercise 11: If $z = 1 + it$ find z^2. Hence sketch the locus of z^2 as t runs from $-\infty$ to ∞.

Exercise 12: Sketch the locus of z^2 as z moves anticlockwise around the square with vertices:

$$0, 1, 1 + i, i.$$

Exercise 13: If $c = \cos\theta$ and $s = \sin\theta$, expand $(c + is)^5$ both by the Binomial Theorem and by de Moivre's Theorem. Substitute $\theta = \pi/5$ and hence find $\sin(\pi/5)$ as a surd.

Exercise 14: Express each of $\sin 7\theta$ and $\cos 7\theta$ in terms of $s = \sin\theta$ and $c = \cos\theta$. Hence express $\tan 7\theta$ in terms of $t = \tan\theta$. Substitute $\theta = \pi/28$ and find a polynomial of degree 7, with integer coefficients, of which $\tan(\pi/28)$ is a zero.

Exercise 15: Prove that if $\text{cis}\theta \neq 1$ then the real part of $(1 - \text{cis}\theta)^{-1}$ is ½.

Exercise 16: Prove that if $\text{cis}\theta \neq 1$ the imaginary part of $(1 - \text{cis}\theta)^{-1}$ is ½ $\cot(\theta/2)$.

Exercise 17: Find the sum of the geometric progression:
$1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{(n-1)i\theta}$.
By equating real and imaginary parts show that

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta = \frac{\cos\left(\frac{n-1}{2}\right)\theta \, \sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}}$$ and

$$1 + \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta = \frac{\sin\left(\frac{n-1}{2}\right)\theta \, \sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}}.$$

Exercise 18: Prove that if z_1 and z_2 are complex numbers then $|z_1 z_2| \geq \mathrm{Re}(z_1\overline{z_2})$.

Exercise 19: Prove that the zeros of
$z^2 + (2\mathrm{cis}\theta)z + \cos 2\theta$
are $z = -\mathrm{cis}\theta \pm (1 + \mathrm{i})\sqrt{\frac{\sin 2\theta}{2}}$.

Exercise 20: Prove that for all positive integers m and all non-real complex numbers z, the expression

$\mathrm{E} = \dfrac{z^m - (\overline{z})^m}{z - \overline{z}}$ is real.

Exercise 21: Prove that if a and b are complex numbers and z is a zero of the polynomial $z^2 + 2az + b^2$ then

$$|z| \leq 2|a| + |b|.$$

Exercise 22: Prove that if a and b are complex numbers and both zeros of the quadratic equation $z^2 + 2az + b^2 = 0$ have modulus ≤ 1 then $|a| \leq 1$ and $|b| \leq 1$.
Is the converse true?

SOLUTIONS FOR CHAPTER 8

Exercise 1: (a) $(2 + 3i) + (5 - 7i) = 7 - 4i$.
(b) $(1 + 3i)(4 - 7i) = 4 + 5i - 21i^2 = 4 + 5i + 21 = 25 + 5i$.
(c) $\dfrac{1}{1 + 3i} = \dfrac{1 - 3i}{(1 + 3i)(1 - 3i)} = \dfrac{1 - 3i}{10} = \dfrac{1}{10} - \dfrac{3}{10}i$.
(d) $\dfrac{1 - 3i}{1 + 3i} = \dfrac{(1 - 3i)^2}{10} = \dfrac{-8 - 6i}{10} = -\dfrac{4}{5} - \dfrac{3}{5}i$.

Exercise 2: (a) $(3 - 7i) - (1 + 2i) = 4 - 9i$.
(b) $(1 + 3i)^2 = 1 + 6i + 9i^2 = -8 + 6i$.
(c) $\dfrac{1}{2 - 3i} = \dfrac{2 + 3i}{13} = \dfrac{2}{13} + \dfrac{3}{13}i$.
(d) $\dfrac{1 + i}{1 - i} = \dfrac{2i}{2} = i$.

Exercise 3: (i) $-4i = 4\text{cis}(3\pi/2) = 4e^{3\pi/2}$.
(ii) Let $z = r\ \text{cis}\theta$. Then $z^2 = r^2\ \text{cis}2\theta = 4\ \text{cis}(3\pi/2)$.
Hence $r^2 = 4$ and $2\theta = 3\pi/2 + 2k\pi$, for some k.
Thus $r = 2$ and $\theta = 3\pi/4 + k\pi$.
Thus $\text{cis}\theta = \text{cis}(3\pi/4)$ or $\text{cis}(7\pi/4)$.
Hence $z = 2\text{cis}(3\pi/4) = 2\left(\dfrac{-1 + i}{\sqrt{2}}\right)$ or

$2\text{cis}(7\pi/4) = 2\left(\dfrac{1 - i}{\sqrt{2}}\right)$

$= \pm(-\sqrt{2} + \sqrt{2}\,i)$.

(iii) $z = \dfrac{1 + i \pm \sqrt{(1 + i)^2 - 6i}}{2}$

$= \dfrac{1 + i \pm \sqrt{-4i}}{2}$

$= \dfrac{1 + i \pm (-\sqrt{2} + \sqrt{2}i)}{2}$

$= \dfrac{(1 - \sqrt{2})}{2} + \dfrac{(1 + \sqrt{2})}{2} i$ or $\dfrac{(1 + \sqrt{2})}{2} + \dfrac{(1 - \sqrt{2})}{2} i$.

Exercise 4: $z = \dfrac{1 + \sqrt{3}i \pm \sqrt{(1 + \sqrt{3}i)^2 + 4}}{2}$

$= \dfrac{1 + \sqrt{3}i \pm \sqrt{-2 + 2\sqrt{3}i}}{2}$.

If $-2 + 2\sqrt{3}\,i = r\,\text{cis}\theta$ then $r = |-2 + 2\sqrt{3}\,i\,| = 4$, $\cos\theta = -\dfrac{1}{2}$ and $\sin\theta = \dfrac{\sqrt{3}}{2}$.

So take $\theta = -\dfrac{\pi}{3}$.

The square roots of $-2 + 2\sqrt{3}\,i$ are thus $\pm\, 2\text{cis}(-\pi/6)$

$= \pm\,(-\sqrt{3} + i)$.

Hence $z = \dfrac{1 + \sqrt{3}i \pm (-\sqrt{3} + i)}{2}$

$= \dfrac{1 - \sqrt{3}}{2} + \dfrac{1 + \sqrt{3}}{2} i$ or $\dfrac{1 + \sqrt{3}}{2} + \dfrac{-1 + \sqrt{3}}{2} i$.

Exercise 5: (i) $Z = \pm 1, \pm i$.

(ii) So $\frac{z-1}{z} = \pm 1, \pm i$.

If $\frac{z-1}{z} = 1$ we get a contradiction.

If $\frac{z-1}{z} = -1$ we get $z = \frac{1}{2}$.

If $\frac{z-1}{z} = i$ we get $z = \frac{1}{1-i} = \frac{1+i}{2}$.

If $\frac{z-1}{z} = -i$ we get $z = \frac{1}{1+i} = \frac{1-i}{2}$.

Hence the three solutions are $\frac{1}{2}$ and $\frac{1 \pm i}{2}$.

(iii) If $4z^3 - 6z^2 + 4z - 1 = 0$ then
$z^4 - 4z^3 + 6z^2 - 4z + 1 = z^4$ and so
$(z-1)^4 = z^4$. This is an equivalent equation to the one in (ii).

Exercise 6: $\left(\frac{z+i}{2z-i}\right)^3 = 1$.

Hence $\frac{z+i}{2z-i} = 1, \omega$ or ω^2, where

$$\omega = \text{cis}(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

If $\frac{z+i}{2z-i} = 1$ then $z = 2i$.

If $\frac{z+i}{2z-i} = \omega$ then $z = \frac{1+\omega}{2\omega-1}$

$$= \frac{1 + \sqrt{3}}{-2 + \sqrt{3}i} = \frac{(1 + \sqrt{3}i)(-2 - \sqrt{3}i)}{7} = \frac{1 - 3\sqrt{3}i}{7}.$$

If $\frac{z + i}{2z - i} = \omega^2$ then $z = \frac{1 + 3\sqrt{3}i}{7}$.

So the three solutions are 2i, $\frac{1 \pm 3\sqrt{3}i}{7}$.

Exercise 7:

$2(1 + 2i)^3 - 7(1 + 2i)^2 + 16(1 + 2i) - 15$

$= 2(1 + 6i - 12 - 8i) - 7(-3 + 4i) + 16(1 + 2i) - 15$

$= 0.$

Hence z = 1 – 2i is also a zero.

The quadratic with these as roots is $z^2 -2z + 5$. (Note this is $z^2 - 2\text{Re}(1 + 2i)z + |1 + 2i|^2$.

Dividing $z^2 - 2z + 5$ into $2z^3 - 7z^2 + 16z - 15$ we find the other factor to be 2z – 3. Hence 3/2 is the remaining zero.

Exercise 8:

$(1 - i)^4 - (1 - i)^3 + (1 - i)^2 + 2 = (1 - 4i + 6i^2 - 4i^3 + i^4) - (1 - 3i + 3i^2 - i^3) + (1 - 2i + i^2) + 2$

$= - 4 - (- 2 - 2i) + (- 2i) + 2$

$= 0.$

Hence 1 + i is also a zero,

The quadratic with 1 ± i as its zeros is $z^2 - 2z + 2$.

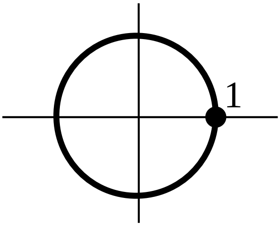
Dividing $z^4 - z^3 + z^2 + 2$ by $z^2 - 2z + 2$ we get $z^2 + z + 1$ as the other factor.

The zeros of $z^2 + z + 1$ are $\frac{-1 \pm \sqrt{3}i}{2}$.

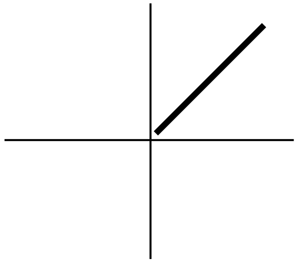
Hence the other zeros are 1 + i and $\frac{-1 \pm \sqrt{3}i}{2}$.

Exercise 9:

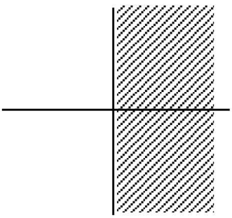
(a)



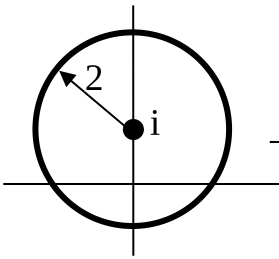
(b)



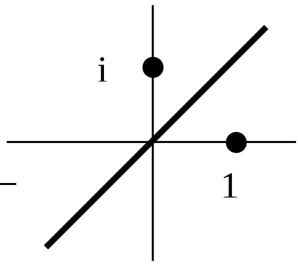
(c)



(d)

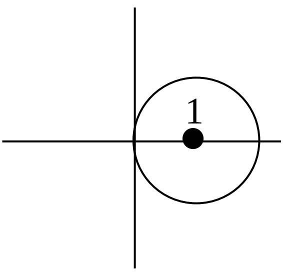


(e)

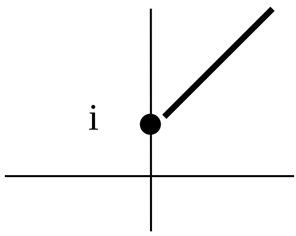


Exercise 10:

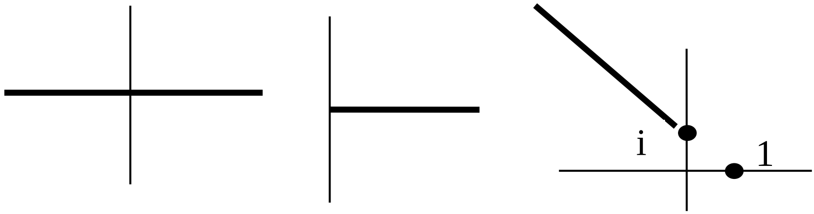
(a)



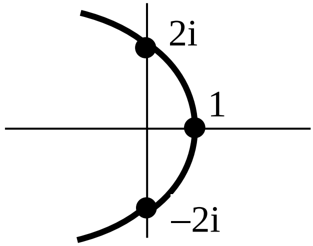
(b)



(c) (d) (e)



Exercise 11: If $z = 1 + it$ then
$z^2 = (1 - t^2) + 2it = x + iy$ where $x = 1 - t^2$ and $y = 2t$.
Since $y^2 + 4x = 4$ the locus is a parabola.

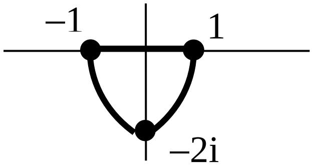


Exercise 12: From $z = 0$ to $z = 1$, z^2 moves along the x-axis from 0 to 1.

As z moves from $z = 1$ to $z = 1 + i$, $z = 1 + it$ for $t = 0$ to 1 and, as in Exercise 11, z^2 moves along a parabola, to – 2i.

As z moves from 1 + i to i, z^2 moves along the mirror image of this parabola from – 2i to i.

Then, as z moves from i back to 0, z^2 moves from –1 to 0 along the real axis.



Exercise 13: $(c + is)^5$
$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$
$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$
$= \cos 5\theta + i \sin 5\theta$ by de Moivre's Theorem.
Hence $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$.
Put $\theta = \pi/5$.
Then $5c^4s - 10c^2s^3 + s^5 = \sin \pi = 0$.
Since $s = \sin(\pi/5) \neq 0$, $5c^4 - 10c^2s^2 + s^4 = 0$.
Substituting $c^2 = 1 - s^2$ we get:

$$5(1 - s^2)^2 - 10(1 - s^2)s^2 + s^4 = 0.$$

Hence $16s^4 - 20s^2 + 5 = 0$.
This is a quadratic in s^2 so $s^2 = \dfrac{20 \pm \sqrt{80}}{32}$ and hence
$s = \pm\sqrt{\dfrac{20 \pm \sqrt{80}}{32}}$.
By checking the approximate values of this expression we find that $\sin(\pi/5) = \sqrt{\dfrac{20 - \sqrt{80}}{32}}$.

Exercise 14: $(c + is)^7$
$= c^7 + 7ic^6s - 21c^5s^2 - 35ic^4s^3 + 35c^3s^4$
$+ 21ic^2s^5 - 7cs^6 - is^7$

$= (c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6)$
$\qquad + i(7c^6s - 35c^4s^3 + 21c^3s^5 - s^7)$
= cos 7θ + i sin 7θ by de Moivre's Theorem.
Hence $\sin 7\theta = 7c^6s - 35c^4s^3 + 21c^3s^5 - s^7$ and
$\cos 7\theta = c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6$.

Hence $\tan 7\theta = \dfrac{7c^6s - 35c^4s^3 + 21c^3s^5 - s^7}{c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6}$.

Dividing top and bottom by c^7 we get

$$\tan 7\theta = \frac{7t - 35t^3 + 21t^5 - t^7}{1 - 21t^2 + 35t^4 - 7t^6}.$$

Put θ = π/28. Then tan 7θ = tan(π/4) = 1 and so, if $t = \tan(\pi/28)$:
$7t - 35t^3 + 21t^5 - t^7 = 1 - 21t^2 + 35t^4 - 7t^6$ and so
$t^7 - 7t^6 - 21t^5 + 35t^4 + 35\,t^3 - 21t^2 - 7t + 1 = 0$.

Exercise 15: Suppose cisθ ≠ 1. Then

$$\frac{1}{1 - \text{cis}\theta} = \frac{1}{1 - \cos\theta + i\sin\theta}$$
$$= \frac{(1 - \cos\theta) - i\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$
$$= \frac{1 - \cos\theta - i\sin\theta}{2(1 - \cos\theta)}.$$

Hence the real part is ½.

Exercise 16: From exercise 15 the imaginary part of $(1 - \text{cis}\theta)^{-1}$ is $-\dfrac{\sin\theta}{2(1 - \cos\theta)}$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{4\cos^2(\theta/2)} = ½ \cot(\theta/2).$$

Exercise 17: The series is a GP with common ratio $e^{i\theta}$.

So $1 + e^{i\theta} + e^{2i\theta} + \ldots + e^{(n-1)i\theta}$

$$= \frac{1 - e^{in\theta}}{1 - e^{i\theta}}$$

$$= \frac{1 - \cos(n\theta) - i\sin(n\theta)}{1 - \cos\theta - i\sin\theta}$$

$$= ½ [1 - \cos(n\theta) - i\sin(n\theta)].[1 + i\cot(\theta/2)].$$

Equating real parts we get

$1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta$

$$= ½ [1 - \cos(n\theta) + \sin(n\theta)\cot(\theta/2)]$$

$$= \sin^2(n\theta/2) + \sin(n\theta/2)\cos(n\theta/2)\cot(\theta/2)$$

$$= \sin^2(n\theta/2) + \sin(n\theta/2)\cos(n\theta/2)\cos(\theta/2)/\sin(\theta/2)$$

$$= \frac{\sin^2(n\theta/2)\sin(\theta/2) + \sin(n\theta/2)\cos(n\theta/2)\cos(\theta/2)}{\sin(\theta/2)}$$

$$= \frac{\sin(n\theta/2)(\sin(n\theta/2)\sin(\theta/2) + \cos(n\theta/2)\cos(\theta/2))}{\sin(\theta/2)}$$

$$= \frac{\sin\left(\frac{n-1}{2}\right)\theta \sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}}.$$

The remaining part is done similarly.

Exercise 18: You probably solved this by letting $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. If so, it probably took you half a page. But here is a one-line solution. Often you get simpler proofs by not splitting complex numbers into their real and imaginary parts.

$$|z_1 z_2| = |z_1|.|z_2| = |z_1|.|\overline{z_2}| = |z_1\overline{z_2}| \geq \mathrm{Re}(z_1\overline{z_2}).$$

Exercise 19: The zeros are:

$$- \mathrm{cis}\theta \pm \sqrt{\mathrm{cis}\ 2\theta - \cos 2\theta}$$
$$= - \mathrm{cis}\ \theta \pm \sqrt{\mathrm{i} \sin 2\theta}$$
$$= - \mathrm{cis}\ \theta \pm \sqrt{\mathrm{cis}(\pi/2) \sin 2\theta}$$
$$= - \mathrm{cis}\ \theta \pm \frac{1 + \mathrm{i}}{\sqrt{2}} \sqrt{\sin 2\theta}\ .$$

Exercise 20: The conjugate of E is clearly E so E is real.

Exercise 21: The zeros are $z = -a \pm \sqrt{a^2 - b^2}$ so
$|z| \leq |-a| + |\sqrt{a^2 - b^2}| = |a| + |\sqrt{a^2 - b^2}|$.
Now $|a^2 - b^2| \leq |a^2| + |-b^2| = |a|^2 + |b|^2 \leq (|a| + |b|)^2$.
Hence $|\sqrt{a^2 - b^2}| \leq |a| + |b|$. Thus $|z| \leq 2|a| + |b|$.

Exercise 22:
If α, β are the roots then $\alpha + \beta = -2a$ and $\alpha\beta = b^2$.
Now $|\alpha + \beta| \leq |\alpha| + |\beta| \leq 2$ and $|\alpha\beta| = |\alpha|.|\beta| \leq 1$.
Hence $|a| \leq 1$ and $|b| \leq 1$.
The converse is not true since the zeros of the polynomial $z^2 - 2z$ are 0, 2.